A New Interval Numbers Power Average Operator in Multiple Attribute Decision Making

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How to fuse uncertain information in multiple attribute decision making (MADM) efficiently is still an open issue. The power average operation is an effective tool to aggregate interval data. However, existing methods to aggregate interval numbers based on power average operator are relatively complicated. In this paper, a simple and effective support function of interval data is proposed. Then, a novel interval number power average operation operator is presented. Finally, a practical MADM problem is used to show the efficiency of the developed method. © 2016 Wiley Periodicals, Inc.

1. INTRODUCTION

Data fusion process is inevitable in some real applications. How to aggregate the given data efficiently and obtain the final decision properly has been discussed heatedly.[1] Many math models such as ordered weighted averaging (OWA) operator,[2–5] power average operator,[6] and fuzzy sets[7] have been proposed and applied to risk assessment,[8–11] decision making,[12–17] uncertain measurement,[18,19] and optimization.[20,21]

It is difficult to give merely clear numbers under the influence of complexity of the object things, limitation of our cognition, and fuzziness of human minds. As a result, interval number is usually used to give possible range of the value. It is an effective tool to show the uncertain information in multiple attribute decision making (MADM). Thus, it is used in MADM problems frequently.[22–25]

Yager[6] introduced the power average operator with the consideration of the relationship between the values to be aggregated. It has some preeminent benefits by a function that indicates the degree values support each other, which makes the operator widely used in many application systems.[26–30] In the aggregation process, the support of other argument values functions as a sort of weight. Many suggested forms

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INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, VOL. 32, 631–644 (2017)

View this article online at wileyonlinelibrary.com. • DOI 10.1002/int.21861

for the support function were given to handle different occasions. Many new types of operators to deal with uncertain values based on the power average operator have been proposed such as intuitionistic fuzzy power aggregation (IFPWA) operator,[31] power generalized interval-valued intuitionistic fuzzy ordered weighted averaging operator and geometric operator[32] and a series of interval-valued Atanassovs IFPWA operators applied to interval-valued Atanassovs intuitionistic fuzzy environments.[33]

However, existing methods to aggregate interval numbers based on power average operator are too complicated to some degree, which may limit the wide use of this operator. To make it more convenient to be applied to practical problems, in this paper, the new interval numbers power average operator is proposed to deal with MADM problems presented by interval numbers.

The rest of the paper is arranged as follows. Section 2 briefly introduces the basic theories such as the power average operator and the related definitions of interval numbers. The new interval numbers power average operator in MADM is presented in Section 3. A practical MADM problem is discussed in Section 4. Finally, this paper is concluded in Section 5.

2. PRELIMINARIES

In this section, some basic concepts and definitions related to interval numbers including definitions and operational laws of interval numbers, the distance between two interval numbers, and the power average operator are introduced, which will be utilized in the latter analysis of the new interval number power average operator.

2.1. Interval Numbers

DEFINITION 2.1.. *Let* $A(a_1, a_2) = \{x | a_1 \leq x \leq a_2\}$, *then we call* $A(a_1, a_2)$ *an interval number. Specifically, when* $a_1 = a_2$, $A(a_1, a_2)$ *is degenerated to a real number.*[34]

Let [**R**] be the set of interval numbers, $A(a_1, a_2)$ and $B(b_1, b_2)$ are two interval numbers in [**R**], where $a_1 \leq a_2, b_1 \leq b_2$, then the operations are as follows:

- $A + B = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$;
- $A - B = (a_1, a_2) - (b_1, b_2) = (a_1 - b_1, a_2 - b_2)$;
- $\lambda A = \lambda(a_1, a_2) = (\lambda a_1, \lambda a_2)$;
- $A = B$ *only if* $a_1 = b_1, a_2 = b_2$.

DEFINITION 2.2.. *Let* $A(a_1, a_2)$ *and* $B(b_1, b_2)$ $(a_1 \leq a_2, b_1 \leq b_2)$ *are two interval numbers in [**R**] and let* $l_a = a_2 - a_1, l_b = b_2 - b_1$, *then the degree of possibility of* $A \geq B$ *is defined as:*[35,36]

$$p(A \geq B) = \max\left\{1 - \max\left(\frac{b_2 - a_1}{l_a + l_b}, 0\right), 0\right\} \tag{1}$$

Similarly, the degree of possibility of $B \geq A$ *is defined as:*

$$p(B \geq A) = \max\left\{1 - \max\left(\frac{a_2 - b_1}{l_a + l_b}, 0\right), 0\right\} \tag{2}$$

Then, we have:

- $0 \leq p(A \geq B) \leq 1$;
- $p(A \geq B) + p(A \leq B) = 1$;
- $p(A \geq B) = 0.5$ only if $a_1 + a_2 = b_1 + b_2$;
- $p(A \geq B) = 1$ only if $a_1 \geq b_2$ and $p(A \geq B) = 0$ only if $a_2 \leq b_1$;
- Let A, B and C be in [**R**], if $p(A \geq B) \geq 0.5$, $p(B \geq C) \geq 0.5$, then $p(A \geq C) \geq 0.5$.

To describe explicitly, we let $p_{ij} = p\ (A_i \geq A_j)$, a complementary matrix is constructed as follows:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \tag{3}$$

It can be easily seen that $\mathrm{p}_{ij} \geq 0, \mathrm{p}_{ij} + \mathrm{p}_{ji} = 1, \mathrm{p}_{ii} = 0.5$ $(i, j \in N$ and $1 \leq i, j \leq n)$.

Then, p_i is defined as the sum of all elements in line i:

$$p_i = \sum_{j=1}^{n} p_{ij} \tag{4}$$

At last, the interval numbers are ranked according to the values of $p_i, i \in N$ in descending order.

2.2. Distance Measure for Interval Numbers

Interval numbers are usually applied to express uncertain evaluation of estimate. Considering every point of both intervals, Tran and Duckstein[37] presented an efficient distance measure to construct the distance between two interval numbers.

DEFINITION 2.3. *Let $A(a_1, a_2)$ and $B(b_1, b_2)$ $(a_1 \leq a_2, b_1 \leq b_2)$ are two interval numbers in [**R**], the distance between A and B is defined as follows:*[37]

$$D^2(A, B) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \left\{ \left[\left(\frac{a_1 + a_2}{2} \right) + x(a_2 - a_1) \right] \right.$$

$$\left. - \left[\left(\frac{b_1 + b_2}{2} \right) + y(b_2 - b_1) \right] \right\}^2 dxdy \tag{5}$$

$$= \left[\left(\frac{a_1 + a_2}{2} \right) - \left(\frac{b_1 + b_2}{2} \right) \right]^2 + \frac{1}{3} \left[\left(\frac{a_2 - a_1}{2} \right)^2 + \left(\frac{b_2 - b_1}{2} \right)^2 \right] \tag{6}$$

It can be easily proved that the distance has these properties:

- Nonnegativity: $d(A, B) \geq 0$;
- Symmetry: $d(A, B) = d(B, A)$;
- Homogeneity: $d(kA, kB) = |k|\, d(A, B)$;
- Translation invariance: $d(A + C, B + C) = d(A, B)$;
- Triangle inequality: $\mathrm{d(A, B)} \leq \mathrm{(A, C)} + \mathrm{d(B, C)}$.

2.3. Power Average Operator

Yager[6] introduced the concept of the power average operation as an aggregation operation, via which a single value can be obtained from a collection of values.

Definition 2.4. *Let $(a_1, a_2, \ldots, a_n)$ be a collection of data; to provide more versatility in the data aggregating process, the power average operator is defined as follows:*[6]

$$P - A(a_1, a_2, ..., a_n) = \frac{\sum_{i=1}^{n} (1 + T(a_i))a_i}{\sum_{i=1}^{n} 1(1 + T(a_i))} \tag{7}$$

where

$$T(a_i) = \sum_{\substack{\mathrm{j=1} \\ \mathrm{j \neq i}}}^{n} \mathrm{Sup}(a_i, a_j) \tag{8}$$

$\mathrm{Sup}(A_i, A_j)$ is denoted as the support for A_i from A_j, thus $T(A_i)$ is the total support for A_i from all the values except for itself.

When the support function is defined, the following properties should be satisfied:

- $\mathrm{Sup}(a, b) = \mathrm{Sup}(b, a)$;
- $\mathrm{Sup}(a, b) \in [0, 1]$;
- $\mathrm{Sup}(a, b) \geq \mathrm{Sup}(x, y)\ if\ \ |a - b| < |x - y|$.

Thus, the closer two values are, the more they support each other.

For convenience, we can denote a vote function:

$$V_i = 1 + T(a_i) \tag{9}$$

The weight of a_i can be obtained as follows:

$$\omega_i = V_i \Big/ \sum_{i=1}^{n} V_i \tag{10}$$

The power average operator can be denoted as follows:

$$P - A = \sum_{i=1}^{n} \omega_i * a_i \tag{11}$$

It is a nonlinear weighted average of the a_i that makes it possible to take the relationship between the values into consideration and the order of them has no influence on the final result for the commutativity of the formula. In addition, if the support function is identically equal to 0, the power average operator degenerates to the simple average.

The power average operator has many advantages as follows:

- Boundedness: $\min\{a_i\} \leq P - A(a_1, a_{2,} \ldots, a_n) \leq \max\{a_i\}$;
- Idempotency: $P - A(a_1, a_{2,} \ldots, a_n) = a$ if $a_i = a$ for all i;
- Commutativity: Any permutation of the arguments has the same power average;
- Nonmonotonous: An increase in one of the arguments can result in a decrease in the power average.

3. A NEW INTERVAL NUMBERS POWER AVERAGE OPERATOR IN MADM

In this section, we propose a new interval numbers power average operator and show its application in MADM.

3.1. Proposed Interval Numbers Power Average Operator

DEFINITION 3.1. *Let $\{A_1, A_2, ..., A_n\}$ be a collection of interval numbers. To provide a single interval number with $\{A_1, A_2, ..., A_n\}$, the proposed interval numbers power average operator is defined as follows:*

$$P - A = \frac{\sum\limits_{i=1}^{n} (1 + T(A_i))A_i}{\sum\limits_{i=1}^{n} 1(1 + T(A_i))} \tag{12}$$

where

$$T(A_i) = \sum_{\substack{\mathrm{j=1} \\ \mathrm{j \neq i}}}^{n} \mathrm{Sup}(\tilde{A}_i, \tilde{A}_j) \tag{13}$$

Here, $\{\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n\}$ are parts of the unit interval $[0, 1]$. We normalize $\{A_1, A_2, ..., A_n\}$ by mapping from them to the unit interval to keep the value of the support function $\mathrm{Sup}(\tilde{A}_i, \tilde{A}_j)$, which reflects the support degree of the two interval numbers A_i, A_j between 0 and 1.

Similarly, when the support function is defined, the following properties should be satisfied:

- $\mathrm{Sup}(\tilde{A}_i, \tilde{A}_j) = \mathrm{Sup}(\tilde{A}_j, \tilde{A}_i)$;
- $\mathrm{Sup}(\tilde{A}_i, \tilde{A}_j) \in [0, 1]$;
- $\mathrm{Sup}(\tilde{A}_{i1}, \tilde{A}_{j1}) \geq \mathrm{Sup}(\tilde{A}_{i2}, \tilde{A}_{j2})$ if the distance between $\tilde{A}_{i1}$ and $\tilde{A}_{j1}$ is bigger than that between $\tilde{A}_{i2}$ and $\tilde{A}_{j2}$.

Although $\tilde{A}_i, \tilde{A}_j$ are interval numbers, the $\mathrm{Sup}(\tilde{A}_i, \tilde{A}_j)$ should also gather to a clear number as a denotation of the degree that $\tilde{A}_i$ supports $\tilde{A}_j$. It can be easily seen that the preeminent properties in power average operator are reserved completely and it has the advantage to handle issues when the data are a compound of clear numbers and interval numbers, or even just a collection of interval numbers.

Similar to Yager's power average operator,[6] the proposed interval numbers power average operator has some basic properties as follows:

1. Boundedness: $\min(A_i) \leq P - A(A_1, A_2, \ldots, A_n) \leq \max(A_i)$;
2. Idempotency: $P - A(A_1, A_2, \ldots, A_n) = A$ when $A_i \equiv A$;
3. Commutativity: aggregating to the same result regarding of the sorting of $\tilde{A}_i$;
4. Nonmonotonous: Different from the usual average, even if all $\tilde{A}_i \geq \tilde{B}_i$, the comparison between two power averages of them may be $P - A(A_1, A_2, \cdots, A_n) \leq P - A(B_1, B_2, \cdots, B_n)$. This is a manifestation of discount outliers: when an interval number is too far from others, it will be discounted by the decreasing weight.

We can also define the vote function as follows:

$$V_i = 1 + T(A_i) \tag{14}$$

The weight of A_i can be obtained as follows:

$$\omega_i = V_i \Big/ \sum_{i=1}^{n} V_i \tag{15}$$

Then, the interval numbers power average operator is as follows:

$$P - A = \sum_{i=1}^{n} \omega_i * A_i \tag{16}$$

In addition, if $\mathrm{Sup}(\tilde{A}_i, \tilde{A}_j) = k$ for all $\tilde{A}_i$ and $\tilde{A}_j$, then $T(A_i) = k(n-1)$ for all i and we have

$$P - A(A_1, A_2, \cdots, A_n) = \frac{1}{n} \sum_{i=1}^{n} A_n \tag{17}$$

which means that the proposed interval numbers power average is simply the average of all A_i with the same support element.

3.2. The Math Model in Multiple Attribute Group Decision Making

It is difficult to give merely clear numbers when dealing with the MADM. In this section, how to use the method properly in MADM with interval numbers is discussed. The crucial point is to define the support function rationally.

Step 1. Assume that a committee of l experts $(e_1, e_2, \ldots, e_k, \ldots, e_l)$. In general, a MADM problem can be concisely expressed in l matrixes. The assessment matrix of expert e_k is as follows:[38]

$$e_k = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{array}{c} G_1 \\ G_2 \\ \vdots \\ G_m \end{array} & \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1m} & A_{2m} & \cdots & A_{nm} \end{bmatrix} \end{array} \tag{18}$$

where $x_1, x_2, \ldots, x_n$ are possible alternatives, $G_1, G_2, \ldots, G_m$ are attributes with which performance of alternatives are measured and A_{ij} is the rating of alternative x_i in regard to attribute G_j. In this paper, the rating A_{ij} of alternative x_i is in the form of interval numbers and both weights of attributes and experts are clear numbers.

Step 2. We map from $\{A_{11}, A_{12}, ..., A_{nm}\}$ to the unit interval $[0, 1]$ to get $\{\tilde{A}_{11}, \tilde{A}_{12}, ..., \tilde{A}_{nm}\}$, which are needed in the support function. For alternative x_i $(i = 1, 2, \ldots, n)$, the $d_{j_1 j_2}$ is denoted as the distance between A_{ij_1} and A_{ij_2}. Then, a distance measure matrix D_i is constructed:

$$D_i = \begin{bmatrix} 0 & d_{12} & \cdots & d_{1m} \\ d_{21} & 0 & \cdots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \cdots & 0 \end{bmatrix} \tag{19}$$

Step 3. Here, the following equation is used to calculate to what degree the interval numbers support each other:

$$\mathrm{Sup}(\tilde{A}_{ij_1}, \tilde{A}_{ij_2}) = 1 - d(\tilde{A}_{ij_1}, \tilde{A}_{ij_2}) \tag{20}$$

Some properties are as follows:

- $0 \leq \mathrm{Sup}(\tilde{A}_{ij_1}, \tilde{A}_{ij_2}) \leq 1$ and $0 \leq d(\tilde{A}_{ij_1}, \tilde{A}_{ij_2}) \leq 1$
- $d(\tilde{A}_{ij_1}, \tilde{A}_{ij_2}) = 0$ and $\mathrm{Sup}(\tilde{A}_{ij_1}, \tilde{A}_{ij_2}) = 1$ only if $\tilde{A}_{ij_1} = \tilde{A}_{ij_2}$;
- the larger the difference between $\tilde{A}_{ij_1}$ and $\tilde{A}_{ij_2}$, the closer $\mathrm{Sup}(\tilde{A}_{ij_1}, \tilde{A}_{ij_2})$ is to 0.

The support function is utilized in $\{\tilde{A}_{i1}, \tilde{A}_{i2}, ..., \tilde{A}_{in}\}$ for each i column in Equation 18.

After all the degree of support between interval numbers are obtained, we denote $S_{j_1 j_2}$ is denoted as the support degree between A_{ij_1} and A_{ij_2}. A support measure matrix (SMM) can be constructed for each i column, which shows the agreement between the interval numbers assessment for the alternative x_i ($i = 1, 2, \ldots, n$):

$$\mathrm{SMM}_i = \begin{bmatrix} 1 & S_{12} & \cdots & S_{1m} \\ S_{21} & 1 & \cdots & S_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m1} & S_{m2} & \cdots & 1 \end{bmatrix} \tag{21}$$

It can be seen that the elements in the principal diagonal are 1.

Step 4. Equation 13 is used to obtain $T(A_{ij})$, then add all the elements of each row $j(1 \leq j \leq m)$ and the vote function V_{ij} for A_{ij} is obtained as follows:

$$V_{ij} = 1 + T(A_{ij}) \tag{22}$$

Then, the weight of A_{ij} can be obtained by Equation 15:

$$\omega_{ij} = V_{ij} \Big/ \sum_{j=1}^{n} V_{ij} \tag{23}$$

The evaluation of expert k for alternative i can be calculated by Equation 16.

$$P - A_{ki} = \sum_{j=1}^{m} \omega_{ij} * A_{ij} \tag{24}$$

Step 5. The evaluations for alternative x_i of l experts is aggregated the same way as we aggregate the evaluations for the attribute G_j of an expert.

Step 6. Each alternative has its own assessment in the form of an interval number. They are ranked by Equation1–4. Decision will be made in last based on their sorting.

4. APPLICATION

In this section, the interval number power average operation is applied to MADM with uncertain information presented by interval numbers, the practical example used in Wang and Lee[39] (after proper management in Ref. 31) is adopted to illustrate the efficiency of our proposed operator. A comparison between the proposed operator and the IFPWA operator[31] is given.

Let us consider a situation where there are four software alternatives x_i (i = 1,2,3,4) in the candidate list for us to choose from in order to improve work productivity. Four attributes G_j (j = 1,2,3,4) are evaluated by three experts e_k (k = 1,2,3) who act as the decision makers and their weight vector is $\lambda = (0.4, 0.3, 0.3)^T$. Besides, the considered four attributes are as follows:

- G_1: cost saving;
- G_2: contribution to organization performance;
- G_3: effort to transform from current system;
- G_4: outsourcing software developer reliability.

The weighted vector of them is $\theta = (0.30, 0.25, 0.25, 0.20)^T$.

Step 1. Three interval number decision matrices $e_k = (a_{ij}^{(k)})_{4\times 4}$ are given as Equation 25–27.

$$e_1 = \begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \\ G_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad x_1 \quad & \quad x_2 \quad & \quad x_3 \quad & \quad x_4 \quad \end{array} \\ \begin{bmatrix} (0.4, 0.5) & (0.5, 0.6) & (0.2, 0.2) & (0.3, 0.5) \\ (0.5, 0.5) & (0.6, 0.6) & (0.7, 0.7) & (0.6, 0.8) \\ (0.7, 0.7) & (0.2, 0.5) & (0.4, 0.4) & (0.5, 0.9) \\ (0.3, 0.4) & (0.5, 0.7) & (0.5, 0.8) & (0.8, 0.9) \end{bmatrix} \end{array} \tag{25}$$

$$e_2 = \begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \\ G_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad x_1 \quad & \quad x_2 \quad & \quad x_3 \quad & \quad x_4 \quad \end{array} \\ \begin{bmatrix} (0.3, 0.4) & (0.4, 0.7) & (0.1, 0.1) & (0.2, 0.4) \\ (0.3, 0.5) & (0.5, 0.7) & (0.5, 0.8) & (0.7, 0.7) \\ (0.5, 0.8) & (0.2, 0.4) & (0.4, 0.6) & (0.4, 0.8) \\ (0.4, 0.5) & (0.6, 0.6) & (0.4, 0.4) & (0.7, 0.9) \end{bmatrix} \end{array} \tag{26}$$

$$e_3 = \begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \\ G_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad x_1 \quad & \quad x_2 \quad & \quad x_3 \quad & \quad x_4 \quad \end{array} \\ \begin{bmatrix} (0.5, 0.6) & (0.6, 0.6) & (0.3, 0.3) & (0.2, 0.3) \\ (0.5, 0.6) & (0.7, 0.7) & (0.6, 0.6) & (0.5, 0.7) \\ (0.6, 0.8) & (0.3, 0.5) & (0.3, 0.5) & (0.9, 0.9) \\ (0.3, 0.5) & (0.5, 0.5) & (0.6, 0.8) & (0.6, 0.6) \end{bmatrix} \end{array} \tag{27}$$

Step 2. The interval numbers are multiplied by the weights at first. The results are shown in Tables I–III:

Then, the interval numbers are mapped to the unit interval. Since the interval numbers are already parts of the unit interval, $\{\tilde{A}_{11}, \tilde{A}_{12}, ..., \tilde{A}_{ij}, ..., \tilde{A}_{44}\}$ are still the original ones after mapping.

Table I. Weighted evaluation of expert 1.

	x_1	x_2	x_3	x_4
$\theta_1 * G_1$	(0.120, 0.150)	(0.150, 0.180)	(0.060, 0.060)	(0.090, 0.150)
$\theta_2 * G_2$	(0.125, 0.125)	(0.150, 0.150)	(0.175, 0.175)	(0.150, 0.200)
$\theta_3 * G_3$	(0.175, 0.175)	(0.050, 0.125)	(0.100, 0.100)	(0.125, 0.225)
$\theta_4 * G_4$	(0.060, 0.080)	(0.100, 0.140)	(0.100, 0.160)	(0.160, 0.180)

Table II. Weighted evaluation of expert 2.

	x_1	x_2	x_3	x_4
$\theta_1 * G_1$	(0.090, 0.120)	(0.120, 0.210)	(0.030, 0.030)	(0.060, 0.120)
$\theta_2 * G_2$	(0.075, 0.125)	(0.125, 0.175)	(0.125, 0.200)	(0.175, 0.175)
$\theta_3 * G_3$	(0.125, 0.200)	(0.050, 0.100)	(0.100, 0.150)	(0.100, 0.200)
$\theta_4 * G_4$	(0.080, 0.100)	(0.120, 0.120)	(0.080, 0.080)	(0.140, 0.180)

Table III. Weighted evaluation of expert 3.

	x_1	x_2	x_3	x_4
$\theta_1 * G_1$	(0.150, 0.180)	(0.180, 0.180)	(0.090, 0.090)	(0.060, 0.090)
$\theta_2 * G_2$	(0.125, 0.150)	(0.175, 0.175)	(0.150, 0.150)	(0.125, 0.175)
$\theta_3 * G_3$	(0.150, 0.200)	(0.075, 0.125)	(0.075, 0.125)	(0.225, 0.225)
$\theta_4 * G_4$	(0.060, 0.100)	(0.100, 0.100)	(0.120, 0.160)	(0.120, 0.120)

e_1 is taken as the example to show the aggregation process. Four distance measure matrixes $D_1 - D_4$ are constructed as follows:

$$D_1 = \begin{bmatrix} 0.0000 & 0.0132 & 0.0409 & 0.0658 \\ 0.0132 & 0.0000 & 0.0500 & 0.0553 \\ 0.0409 & 0.0500 & 0.0000 & 0.1052 \\ 0.0658 & 0.0553 & 0.1052 & 0.0000 \end{bmatrix} \tag{28}$$

$$D_2 = \begin{bmatrix} 0.0000 & 0.0173 & 0.0809 & 0.0473 \\ 0.0173 & 0.0000 & 0.0661 & 0.0321 \\ 0.0809 & 0.0661 & 0.0000 & 0.0407 \\ 0.0473 & 0.0321 & 0.0407 & 0.0000 \end{bmatrix} \tag{29}$$

$$D_3 = \begin{bmatrix} 0.0000 & 0.1150 & 0.0400 & 0.0721 \\ 0.1150 & 0.0000 & 0.0750 & 0.0482 \\ 0.0400 & 0.0750 & 0.0000 & 0.0346 \\ 0.0721 & 0.0482 & 0.0346 & 0.0000 \end{bmatrix} \tag{30}$$

$$D_4 = \begin{bmatrix} 0.0000 & 0.0594 & 0.0645 & 0.0532 \\ 0.0594 & 0.0000 & 0.0323 & 0.0163 \\ 0.0645 & 0.0323 & 0.0000 & 0.0299 \\ 0.0532 & 0.0163 & 0.0299 & 0.0000 \end{bmatrix} \tag{31}$$

Step 3. Equations 19–21 is used to get the SMM_i $(i = 1, 2, 3, 4)$ for each alternatives. The results of e_1 are shown as $\mathrm{SMM}_1 - \mathrm{SMM}_4$:

$$\mathrm{SMM}_1 = \begin{bmatrix} 1.0000 & 0.9868 & 0.9591 & 0.9342 \\ 0.9868 & 1.0000 & 0.9500 & 0.9447 \\ 0.9591 & 0.9500 & 1.0000 & 0.8948 \\ 0.9342 & 0.9447 & 0.9895 & 1.0000 \end{bmatrix} \tag{32}$$

$$\mathrm{SMM}_2 = \begin{bmatrix} 1.0000 & 0.9827 & 0.9191 & 0.9527 \\ 0.9827 & 1.0000 & 0.9339 & 0.9679 \\ 0.9191 & 0.9339 & 1.0000 & 0.9593 \\ 0.9527 & 0.9679 & 0.9593 & 1.0000 \end{bmatrix} \tag{33}$$

$$\mathrm{SMM}_3 = \begin{bmatrix} 1.0000 & 0.8850 & 0.9600 & 0.9279 \\ 0.8850 & 1.0000 & 0.9250 & 0.9518 \\ 0.9600 & 0.9250 & 1.0000 & 0.9654 \\ 0.9279 & 0.9518 & 0.9654 & 1.0000 \end{bmatrix} \tag{34}$$

$$\mathrm{SMM}_4 = \begin{bmatrix} 1.0000 & 0.9406 & 0.9355 & 0.9468 \\ 0.9406 & 1.0000 & 0.9677 & 0.9837 \\ 0.9355 & 0.9677 & 1.0000 & 0.9701 \\ 0.9468 & 0.9837 & 0.9701 & 1.0000 \end{bmatrix} \tag{35}$$

The process of the evaluations of SMM_i $(i = 1, 2, 3, 4)$ for $e_2 - e_4$ is the same as e_1.

Step 4. Add all the elements of four rows in Equation 32–35, the vote function V_{ij} of A_{ij} in e_1 can be obtained as follows:

$$V_{11} = 1.0000 + 0.9868 + 0.9591 + 0.9342 = 3.8801$$
$$V_{12} = 0.9868 + 1.0000 + 0.9500 + 0.9447 = 3.8815$$
$$V_{13} = 0.9591 + 0.9500 + 1.0000 + 0.8948 = 3.8039$$
$$V_{13} = 0.9342 + 0.9447 + 0.9895 + 1.0000 = 3.8684$$

and

$$V_{21} = 3.8545, \quad V_{22} = 3.8845, \quad V_{23} = 3.8123, \quad V_{24} = 3.8799$$
$$V_{31} = 3.7729, \quad V_{32} = 3.7618, \quad V_{33} = 3.8504, \quad V_{34} = 3.8451$$
$$V_{41} = 3.8229, \quad V_{42} = 3.8920, \quad V_{43} = 3.8733, \quad V_{44} = 3.9006$$

Then, the weight of A_{ij} in e_1 can be obtained as follows:

$$\omega_{11} = \frac{3.8801}{3.8801 + 3.8815 + 3.8039 + 3.8684} = 0.2514$$
$$\omega_{12} = \frac{3.8815}{3.8801 + 3.8815 + 3.8039 + 3.8684} = 0.2515$$
$$\omega_{12} = \frac{3.8039}{3.8801 + 3.8815 + 3.8039 + 3.8684} = 0.2465$$
$$\omega_{14} = \frac{3.8684}{3.8801 + 3.8815 + 3.8039 + 3.8684} = 0.2506$$

and

$$\omega_{21} = 0.2498, \quad \omega_{22} = 0.2517, \quad \omega_{23} = 0.2471, \quad \omega_{24} = 0.2514$$
$$\omega_{31} = 0.2477, \quad \omega_{32} = 0.2470, \quad \omega_{33} = 0.2528, \quad \omega_{34} = 0.2525$$
$$\omega_{41} = 0.2468, \quad \omega_{42} = 0.2513, \quad \omega_{43} = 0.2501, \quad \omega_{44} = 0.2518$$

The evaluations of $e_2 - e_4$ are the same as e_1.

Step 5. Since each alternative has its own scores from three experts, the next step is fusing them with the consideration of weights of the experts. The results are shown as follows (Table IV).

Here, Equations 19–24 are used as well after they are multiplied by the weights. The final evaluations are presented in Table V.

Step 6. Finally, Equations 1–4 are used to order the four ultimately evaluations. We have

$$P = \begin{bmatrix} 0.5000 & 0.4069 & 0.7824 & 0.1811 \\ 0.5931 & 0.5000 & 0.8681 & 0.2706 \\ 0.2176 & 0.1319 & 0.5000 & 0.0000 \\ 0.8189 & 0.7294 & 1.0000 & 0.5000 \end{bmatrix}$$

and

$$p_1 = 0.5000 + 0.4069 + 0.7824 + 0.1811 = 1.8704$$
$$p_2 = 0.5931 + 0.5000 + 0.8681 + 0.2706 = 2.2318$$
$$p_3 = 0.2176 + 0.1319 + 0.5000 + 0.0000 = 0.8495$$
$$p_4 = 0.8189 + 0.7294 + 1.0000 + 0.5000 = 3.0483$$

$p_4 \succ p_2 \succ p_1 \succ p_3$, thus we have $x_4 \succ x_2 \succ x_1 \succ x_3$, which means that the rank order is $(alternative\ x_4) \succ (alternative\ x_2) \succ (alternative\ x_1) \succ (alternative\ x_3)$ and the forth is the best choice. It is coincided with the results of that presented in Ref. 31.

It can be easily seen that the algorithm proposed before is simpler and more convenient than the IFPWA operator[31] and they are equally effective as shown in the practical MADM problem. In addition, its application will not be limited to the MADM and we can use that to deal with many issues presented by interval numbers in an efficient manner.

Table IV. The assessment of three experts.

	x_1	x_2	x_3	x_4
$P - A_1$	(0.1201, 0.1327)	(0.1127, 0.1488)	(0.1086, 0.1238)	(0.1315, 0.1889)
$P - A_2$	(0.0923, 0.1358)	(0.1040, 0.1513)	(0.0839, 0.1151)	(0.1191, 0.1690)
$P - A_3$	(0.1216, 0.1578)	(0.1326, 0.1450)	(0.1087, 0.1313)	(0.1320, 0.1522)

Table V. The final assessment for the alternatives.

	x_1	x_2	x_3	x_4
$\lambda_1 * P - A_1$	(0.0480, 0.0531)	(0.0451, 0.0595)	(0.0434, 0.0495)	(0.0526, 0.0756)
$\lambda_2 * P - A_2$	(0.0277, 0.0407)	(0.0312, 0.0454)	(0.0252, 0.0345)	(0.0357, 0.0507)
$\lambda_3 * P - A_3$	(0.0365, 0.0473)	(0.0398, 0.0435)	(0.0326, 0.0394)	(0.0396, 0.0457)
$P - A$	(0.0374, 0.0470)	(0.0387, 0.0495)	(0.0337, 0.0411)	(0.0426, 0.0573)

5. CONCLUSION

In this paper, a new method to fuse uncertain values denoted by interval numbers is proposed based on the power average operator presented by Yager.[6] It is shown that the new method can deal with interval numbers aggregation in MADM problems in an efficient manner. We take a practical example to illustrate the use of the proposed method. It can be easily applied to other aggregation problem for its advantages such as simpleness of algorithm and flexibility of the support function. In the future, different support functions will be structured to apply this operator to different occasions or fields.

Acknowledgments

The work is partially supported by the National Natural Science Foundation of China (Grant No. 61671384), the Natural Science Basic Research Plan in Shaanxi Province of China (Program No. 2016JM6018), the Fund of SAST (Program No. SAST2016083), and the Seed Foundation of Innovation and Creation for Graduate Students in Northwestern Polytechnical University (Program No. Z2016122).

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